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The Wilson loop CFT: Insertion dimensions and structure constants from wavy lines

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Abstract

We study operator insertions into the 1/2 BPS Wilson loop in $\mathcal{N} = 4$ SYM theory and determine their two-point coefficients, anomalous dimensions and structure constants. The calculation is done for the first few lowest dimension insertions and relies on known results for the expectation value of a smooth Wilson loop. In addition to the particular coefficients that we calculate, our study elucidates the connection between deformations of the line and operator insertions and between the vacuum expectation value of the line and the CFT data of the insertions.

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1 Introduction

One of the most important questions in $\mathcal{N} = 4$ SYM theory, as in any gauge theory is the evaluation of the vacuum expectation value (VEV) of arbitrary Wilson loop operators. Ideally one would like to calculate that for any value of the coupling. Historically, weak coupling was the only feasible regime in which to work, but holography makes the strong coupling regime accessible too. Integrability based techniques and localization have led to arbitrary coupling results in certain cases.

In $\mathcal{N} = 4$ SYM, the expectation value of a smooth Wilson loop, which is finite, is invariant under non-singular conformal transformations. This statement is no longer true for Wilson loops along singular curves, or for Wilson loops with insertions of operators into them. Cusps and operator insertions may give rise to divergences in perturbation theory and consequentially acquire anomalous dimensions [1–7]. The 1/2-BPS straight and circular Wilson loops are the simplest candidates into which to introduce cusps and/or operator insertions. Indeed this problem was studied in [8–10] and a set of boundary thermodynamic Bethe ansatz equations that calculate their spectrum was written down.¹ These equations led to a numerical solution of the quark-antiquark potential in this theory [11].

In this note we study operator insertions into 1/2-BPS Wilson loops from a different perspective. In addition to determining some of their anomalous dimensions, we also find some structure constants.² To do that we employ the correspondence between small deformations of Wilson loops and Wilson loops with operator insertions.

We define the Wilson loop with insertions as

$$W [\mathcal{O}^{(1)}(x(s_1)) \dots \mathcal{O}^{(n)}(x(s_n))] = \frac{1}{N} \text{tr} \mathcal{P} \left[\mathcal{O}^{(1)}(x(s_1)) \dots \mathcal{O}^{(n)}(x(s_n)) e^{\int (i\dot{x}^\mu A_\mu(x(s)) - |\dot{x}| \Phi^1(x(s))) ds} \right]. \quad (1.1)$$

where³ $|\dot{x}| \equiv \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$ and the integration is over the straight line. The expectation value of (1.1) can be viewed as the correlation functions of the operator insertions, so we define the VEV of the Wilson loop to be exactly that (henceforce we identify $s_i = x(s_i)$)

$$\langle\langle \mathcal{O}^{(1)}(s_1) \dots \mathcal{O}^{(n)}(s_n) \rangle\rangle = \langle W [\mathcal{O}^{(1)}(s_1) \dots \mathcal{O}^{(n)}(s_n)] \rangle / \langle W \rangle. \quad (1.2)$$

These correlation functions satisfy the axioms of a CFT.⁴

The key to this is that the residual symmetry preserved by the 1/2-BPS straight line includes an $SO(1, 2)$ group of conformal transformations along the line⁵. The operator insertions can

¹In the case of the cusp, each ray is on its own 1/2-BPS.

²In this note we do not consider cusps.

³We work with spacetime signature $(+ - - -)$. Our conventions are discussed in more detail in Section 2.2 and Section 2.3.

⁴More precisely, this is a defect CFT, and does not have an energy-momentum tensor that is decoupled from the bulk, see, *e.g.*, [12–15].

⁵The case of the circle could be analyzed just as well, and would give essentially the same result. However, the notation in the case of the line is simpler, as the tangent and normal directions are constant.

be classified by representations of this group (or by the full $OSp(2, 2|4)$ preserved by the line). Furthermore one may define primary operators under the conformal group (and under the superconformal group).

The correlation functions (1.2) are constrained in the same manner as in a usual CFT. For example the Wilson loop with two scalar insertions satisfies the Ward identity for dilatation

$$iD \langle\langle \mathcal{O}^{(1)}(s) \mathcal{O}^{(2)}(0) \rangle\rangle = (\Delta_{\mathcal{O}^{(1)}} + \Delta_{\mathcal{O}^{(2)}} + s \partial_s) \langle\langle \mathcal{O}^{(1)}(s) \mathcal{O}^{(2)}(0) \rangle\rangle = 0, \quad (1.3)$$

which is identical in form to that of the two-point function of local operators, solved by

$$\langle\langle \mathcal{O}^{(1)}(s) \mathcal{O}^{(2)}(0) \rangle\rangle = \frac{a_{\mathcal{O}^{(1)}\mathcal{O}^{(2)}}(\lambda)}{s^{\Delta_{\mathcal{O}^{(1)}} + \Delta_{\mathcal{O}^{(2)}}}}, \quad (1.4)$$

where $\Delta_{\mathcal{O}^{(n)}} = \Delta_{\mathcal{O}^{(n)}}(\lambda)$ and we associate dimensions (classical and anomalous) to the insertions. The two-point function of primaries vanishes unless they have the same dimension due to special conformal symmetry. $a_{\mathcal{O}^{(1)}\mathcal{O}^{(2)}}$, which below is denoted as $a_{\mathcal{O}^{(1)}}$, is the coefficient of the two-point function and is well defined, when the operators are related to deformations of the Wilson loop, as they are in our analysis [16].

In a similar way, the three-point function of scalar primary operators satisfies

$$\langle\langle \mathcal{O}^{(1)}(s_1) \mathcal{O}^{(2)}(s_2) \mathcal{O}^{(3)}(s_3) \rangle\rangle = \frac{c_{(123)}(\lambda)}{|s_{12}|^{\Delta_1 + \Delta_2 - \Delta_3} |s_{13}|^{\Delta_1 - \Delta_2 + \Delta_3} |s_{23}|^{-\Delta_1 + \Delta_2 + \Delta_3}}, \quad (1.5)$$

where $c_{(123)}(\lambda)$ is the structure constant and $s_{ij} = s_i - s_j$. The four-point function is

$$\langle\langle \mathcal{O}^{(1)}(s_1) \mathcal{O}^{(2)}(s_2) \mathcal{O}^{(3)}(s_3) \mathcal{O}^{(4)}(s_4) \rangle\rangle = \frac{G_{1234}(u)}{\prod_{i < j} |s_{ij}|^{\Delta_i + \Delta_j - \Delta}}, \quad \Delta = \frac{1}{3} \sum_{i=1}^4 \Delta_i, \quad (1.6)$$

where G_{1234} is a function of the real cross-ratios u . It can in principle be determined from the structure constants via the operator product expansion, but we shall not explore that relation.

Above we have considered scalar primary operators. All of these constraints generalize to descendant operators and to operators with tensor structure. The correlation function of a descendant operator is obtained from the correlation function of the primary operator by the application of the lowering operator of the residual conformal algebra. For example, for the 1/2 BPS line along the x^3 direction, this lowering operator is ∂_3 .

The correlation functions of tensor operators are constructed from the inversion tensor $I_{\mu\nu}(x) = \eta_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2}$ and from the vector $Y^\mu(x_1, x_2, x_3) = \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}$. For insertions along a line, the former reduces in the transverse directions to $I_{ij} = \eta_{ij}$, with $i = 0, 1, 2$ and the latter vanishes $Y^i = 0$. Let us consider a vector primary operator \mathcal{O}_i as an example. Its two-point function is given by

$$\langle\langle \mathcal{O}_i(s) \mathcal{O}_j(0) \rangle\rangle = \frac{a_{\mathcal{O}}(\lambda) \eta_{ij}}{s^{2\Delta_{\mathcal{O}}}}, \quad (1.7)$$

and its three-point function is constrained to vanish. This generalizes straightforwardly to higher rank tensor operators. In this way, as for a CFT, the correlation functions of the insertions are determined up to a set of coefficients, the CFT data.

The anomalous dimensions can be determined in principle at all values of the gauge coupling by using the tools of integrability. Thus far no analogous techniques were developed to understand the three-point functions of insertions. Alternatively one can use Feynman diagrams to evaluate any of those correlation functions. For example, Figure 1 shows one Feynman diagram contributing to the two-point function of insertions at two-loops in the coupling. Note that the correlation function includes interactions between the insertions and the Wilson loop itself.

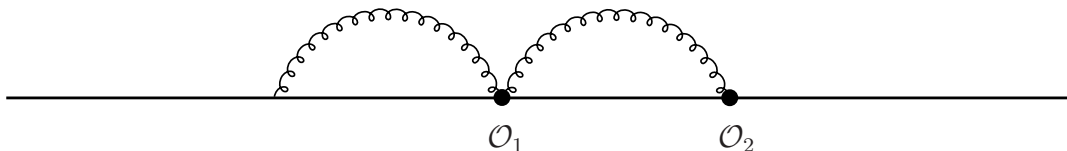


Figure 1: A Feynman diagram contributing to $\langle\langle \mathcal{O}_1 \mathcal{O}_2 \rangle\rangle$. The straight line represents the 1/2 BPS line. The wavy lines represent propagators in the theory, the exact form of which depends on the operators \mathcal{O}_1 and \mathcal{O}_2 .

In this note we take a different approach, based on the calculation of smooth Wilson loops which are small deformations of the straight line. Such deformations can be written as integrals of local operator insertions into the straight line. In the following we classify the low dimension insertions into the straight Wilson line and find the exact mapping between deformations and operator insertions by performing a functional Taylor expansion.

The vacuum expectation value of the nearly straight Wilson loop may be calculated in perturbation theory and in certain instances also in string theory, giving expressions at weak and strong coupling [17]. These calculations should match the appropriate sum of integrated n -point functions of the appropriate insertions. We demonstrate this procedure and extract the first nontrivial anomalous dimensions and structure constants of the lowest dimension insertions.

At one-loop, *i.e.* order $\lambda = g^2 N$, we only have two-point functions of operator insertions contributing to the expectation value of the Wilson loop. More specifically, we have tree-level two-point functions of the lowest dimension operators. At two-loops, *i.e.* order λ^2 , we have two, three and (factorized) four-point functions. The three and four-point functions are tree-level and the two-point functions include both the one-loop corrections to the two-point functions of the lowest dimension insertions and the tree-level correlators of composite operators. As we only have tree-level two-point functions of composite operators, we do not expect operator mixing at this order.

To perform the calculation, we use known compact expressions for the planar vacuum expectation values at one and two-loops at weak coupling. At one-loop there is the usual well-known expression for the expectation value of a general loop. At two-loops the expression for the ex-

pectation value of a general planar contour was found in [18]. We manipulate these expressions into the form of a sum of n -point functions of operator insertions, such that we may read off the coefficients of the n -point functions using the mapping described above.

In [17] the expectation value of a general deformation of the 1/2 BPS straight line in \mathbb{R}^4 was computed as an expansion in the deformation. This was done to second order in the deformation, to two-loop order at weak coupling and at leading order at strong coupling. This computation revealed the so-called ‘universality’ of the deformed line - at second order in the deformation the functional form of the expectation value is the same at one and two-loops, and also at strong coupling. As we outline below, in the operator insertion language the universality observed at second order in the deformation is equivalent to the statement that a specific operator insertion is a protected operator. The coupling dependent two-point coefficient for this protected operator was then understood to be related to Bremsstrahlung radiation of an accelerated quark [16].

The correspondence between operator insertions into the line and deformation of the line was also checked explicitly in [17] at one-loop order via a Feynman diagram computation. In [19] a further Feynman diagram computation was performed to find the anomalous dimension of Φ^1 as an insertion in the 1/2 BPS line coupling to Φ^1 . Our analysis is consistent with the results of [17] and [19]. The results also satisfy several consistency conditions imposed by the multiplet structure of the insertions. We find several two-point coefficients and structure constants which have not been previously calculated.

2 Equivalence of operator insertions and deformations

In this section we review the equivalence between small deformations of a Wilson loop and operator insertions along the loop. The procedure is general and holds for deformations of any smooth loop, however starting from Section 2.2 we restrict ourselves to deformations of the 1/2 BPS straight line.

2.1 The operator expansion

Let us consider a Wilson loop (or line) along a contour \mathcal{C}_0 in $\mathbb{R}^{1,3}$, defined by $x^\mu(s)$ and let \mathcal{C} be a deformation of this loop given by $x^\mu(s) + \delta x^\mu(s)$. The Wilson loop $W[\mathcal{C}]$ can then be written formally by the functional Taylor expansion

$$W[\mathcal{C}] = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n W[\mathcal{C}_0], \quad (2.1)$$

where δ corresponds to taking $x^\mu(s) \rightarrow x^\mu(s) + \delta x^\mu(s)$. We do not consider deformations of the scalar couplings, beyond the change of the magnitude of the coupling of Φ^1 , which is $|\dot{x}|$. This expansion is valid for any value of the coupling.

It is straightforward to take the functional variation of the Wilson loop. Clearly the first order variation $\delta W[\mathcal{C}_0]$ is equivalent to the insertion of an operator $\mathcal{S}^{(1)}(s_1)$ into $W[\mathcal{C}_0]$

$$\delta W[\mathcal{C}_0] = \text{tr} \mathcal{P} \oint ds_1 \mathcal{S}^{(1)}(s_1) e^{i \oint ds \mathbb{A}(s)}, \quad (2.2)$$

where $\mathbb{A}(s)$ is the connection of the undeformed loop $W[\mathcal{C}_0]$ and the integrals are along the contour \mathcal{C}_0 .

The second order variation is equivalent to two insertions of $\mathcal{S}^{(1)}$ in $W[\mathcal{C}_0]$ as well as the insertion of a new operator $\mathcal{S}^{(2)}(s)$ into the loop, *i.e.*

$$\delta^2 W[\mathcal{C}_0] = \text{tr} \mathcal{P} \oint ds_1 \mathcal{S}^{(2)}(s_1) e^{i \oint ds \mathbb{A}(s)} + \text{tr} \mathcal{P} \oint ds_1 ds_2 \mathcal{S}^{(1)}(s_1) \mathcal{S}^{(1)}(s_2) e^{i \oint ds \mathbb{A}(s)}. \quad (2.3)$$

The $\mathcal{S}^{(n)}$ are related by the recursion relation

$$\mathcal{S}^{(n+1)} = \delta \mathcal{S}^{(n)} - i \delta x^\mu [A_\mu, \mathcal{S}^{(n)}] = \left(\delta x^\mu D_\mu + \delta \dot{x}^\mu \frac{\partial}{\partial \dot{x}^\mu} \right) \mathcal{S}^{(n)}. \quad (2.4)$$

We may use the above to relate the expectation values $\langle W[\mathcal{C}_0] \rangle$ and $\langle W[\mathcal{C}] \rangle$. Using the double bracket notation (1.2) we have

$$\frac{\delta \langle W[\mathcal{C}_0] \rangle}{\langle W[\mathcal{C}_0] \rangle} = \oint ds_1 \langle\langle \mathcal{S}^{(1)}(s_1) \rangle\rangle, \quad (2.5)$$

$$\frac{\delta^2 \langle W[\mathcal{C}_0] \rangle}{\langle W[\mathcal{C}_0] \rangle} = \oint ds_1 \langle\langle \mathcal{S}^{(2)}(s_1) \rangle\rangle + \oint ds_1 ds_2 \langle\langle \mathcal{S}^{(1)}(s_1) \mathcal{S}^{(1)}(s_2) \rangle\rangle. \quad (2.6)$$

At higher orders we find

$$\begin{aligned} \frac{\delta^3 \langle W[\mathcal{C}_0] \rangle}{\langle W[\mathcal{C}_0] \rangle} &= \oint ds_1 \langle\langle \mathcal{S}^{(3)}(s_1) \rangle\rangle + \oint ds_1 ds_2 3 \langle\langle \mathcal{S}^{(1)}(s_1) \mathcal{S}^{(2)}(s_2) \rangle\rangle \\ &\quad + \oint ds_1 ds_2 ds_3 \langle\langle \mathcal{S}^{(1)}(s_1) \mathcal{S}^{(1)}(s_2) \mathcal{S}^{(1)}(s_3) \rangle\rangle \end{aligned} \quad (2.7)$$

$$\begin{aligned} \frac{\delta^4 \langle W[\mathcal{C}_0] \rangle}{\langle W[\mathcal{C}_0] \rangle} &= \oint ds_1 \langle\langle \mathcal{S}^{(4)}(s_1) \rangle\rangle + \oint ds_1 ds_2 (4 \langle\langle \mathcal{S}^{(1)}(s_1) \mathcal{S}^{(3)}(s_2) \rangle\rangle + 3 \langle\langle \mathcal{S}^{(2)}(s_1) \mathcal{S}^{(2)}(s_2) \rangle\rangle) \\ &\quad + \oint ds_1 ds_2 ds_3 6 \langle\langle \mathcal{S}^{(1)}(s_1) \mathcal{S}^{(1)}(s_2) \mathcal{S}^{(2)}(s_3) \rangle\rangle \\ &\quad + \oint ds_1 ds_2 ds_3 ds_4 \langle\langle \mathcal{S}^{(1)}(s_1) \mathcal{S}^{(1)}(s_2) \mathcal{S}^{(1)}(s_3) \mathcal{S}^{(1)}(s_4) \rangle\rangle. \end{aligned} \quad (2.8)$$

We restrict our analysis to the fourth order in the deformation.

2.2 Deformations of the straight line and explicit form of the operators

In this section, adopting the conventions of [20–22], we explain the explicit form of the operators $\mathcal{S}^{(n)}$ for general deformations of the 1/2 BPS Wilson line in $\mathbb{R}^{1,3}$ (with signature $(+ - - -)$).⁶ The 1/2 BPS line with spacetime contour $x^\mu(s) = (0, 0, 0, s)$ is given by

$$W_{\text{BPS}} = \text{tr } \mathcal{P} \exp \left(i \oint ds \mathbb{A} \right), \quad \mathbb{A} = A_3 + i\Phi^1. \quad (2.9)$$

The first combination of operators which appears in the expansion about a general contour is (2.5)

$$\mathcal{S}^{(1)} = \delta x^\mu \left(i\dot{x}^\nu F_{\mu\nu} - |\dot{x}| D_\mu \Phi^1 \right) + \frac{\delta \dot{x} \cdot \dot{x}}{|\dot{x}|} \Phi^1, \quad (2.10)$$

recalling that $|\dot{x}| = \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$. The rest of the operators are given by the relation (2.4). Focusing on the straight line, by reparametrization of the loop, we can assume that the deformations take the form $\delta x^\mu(s) = (\epsilon^i(s), 0)$ where $i = 0, 1, 2$. $\epsilon = |\epsilon| \ll 1$ serves as our expansion parameter,⁷ and we assume that $\frac{d^n}{ds^n} \epsilon(s) \sim \epsilon(s)$. To fourth order we then have

$$\begin{aligned} \mathcal{S}^{(1)} &= i\mathbb{F}_{i3}\epsilon^i, \\ \mathcal{S}^{(2)} &= iD_i\mathbb{F}_{j3}\epsilon^i\epsilon^j + iF_{ij}\epsilon^i\dot{\epsilon}^j + \Phi^1\dot{\epsilon}^2, \\ \mathcal{S}^{(3)} &= iD_iD_j\mathbb{F}_{k3}\epsilon^i\epsilon^j\epsilon^k + 2iD_iF_{jk}\epsilon^i\epsilon^j\dot{\epsilon}^k + 3D_i\Phi^1\dot{\epsilon}^2\epsilon^i, \\ \mathcal{S}^{(4)} &= iD_iD_jD_k\mathbb{F}_{m3}\epsilon^i\epsilon^j\epsilon^k\epsilon^m + 3iD_iD_jF_{km}\epsilon^i\epsilon^j\epsilon^k\dot{\epsilon}^m + 6D_iD_j\Phi^1\dot{\epsilon}^2\epsilon^i\epsilon^j + 3\Phi^1\dot{\epsilon}^4. \end{aligned} \quad (2.11)$$

Here we have introduced the notation $\mathbb{F}_{i3} = F_{i3} + iD_i\Phi^1$. Generally, the operators we encounter in this expansion are

$$D_{i_1} \cdots D_{i_n} \mathbb{F}_{i_{n+1}3}, \quad D_{i_1} \cdots D_{i_n} F_{i_{n+1}j}, \quad D_{i_1} \cdots D_{i_n} \Phi^1, \quad (2.12)$$

with $n \geq 0$. All the indices, except for the j index in the expressions above are symmetrized, since these operators always appear contracted to ϵ^i (and $\dot{\epsilon}^j$). The last operator can appear with arbitrary even powers of $\dot{\epsilon}$.

At strong coupling there is an algorithm to study the minimal surface associated to small deformations of the straight Wilson line, as long as the deformations leave the contour \mathcal{C} in \mathbb{R}^2 [23]. In that case the operator F_{ij} and its derivatives do not appear in the expansion.

⁶With this space-time signature, supersymmetry implies also a negative inner product between the scalars, so the propagator is $\langle \Phi^1(x) \Phi^1(0) \rangle = -\frac{1}{4\pi|x|^2}$.

⁷To be more precise, we assume that each component of ϵ is small, *i.e.* $|\epsilon^i| \ll 1$.

2.3 Operators representation and multiplets

The introduction of the 1/2 BPS space-like line breaks the $PSU(2, 2|4)$ symmetry of $\mathcal{N} = 4$ SYM to a $OSp(2, 2|4)$ residual superconformal group [24]. In this section we classify the operator insertions in terms of their irreducible bosonic representations and the supermultiplets they belong to.

The 1/2 BPS line breaks the $SO(1, 3)$ Lorentz group to $SO(1, 2)$. We take the chiral form of the gamma matrices, *i.e.*

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\beta}} \\ \bar{\sigma}^{\mu, \dot{\alpha}\beta} & 0 \end{pmatrix}, \quad (2.13)$$

where we choose $\sigma^\mu = (1, \tau^1, \tau^3, \tau^2)$ and τ^a are the Pauli matrices.⁸ These split according to

$$\sigma^{\mu, \dot{\alpha}\beta} \rightarrow \{\sigma^{i, \alpha\beta}, -i\epsilon^{\alpha\beta}\}, \quad \bar{\sigma}^\mu_{\dot{\alpha}\beta} \rightarrow \{\sigma^i_{\alpha\beta}, i\epsilon_{\alpha\beta}\}. \quad (2.14)$$

Since we now have only one $SU(2)$ (more precisely $SO(1, 2)$), there is no way to distinguish dotted and undotted indices and indeed we can raise and lower them using $\epsilon^{\alpha\beta}$.

The $SO(6)$ R-symmetry also breaks to $SO(5) \simeq Sp(4)$. Taking the $SO(6)$ gamma matrices to be in a chiral form, $\rho^{A, ab}$ and $\bar{\rho}^A_{ab}$, they split as

$$\rho^{I, ab} \rightarrow \{\omega^{ab}, \rho^{A, ab}\}, \quad \bar{\rho}^I_{ab} \rightarrow \{\omega_{ab}, \rho^A_{ab}\}, \quad (2.15)$$

with $A = 2, \dots, 6$. Again we can use the distinguished $\rho^1 = \omega$ to raise and lower indices according to $s^a = \omega^{ab}s_b$, $s_a = s^b\omega_{ba}$ and consequently $\omega^{ab}\omega_{bc} = -\delta^a_c$. Similarly, we raise and lower $SU(2)$ spinor indices as $s^\alpha = \epsilon^{\alpha\beta}s_\beta$ and $s_\alpha = s^\beta\epsilon_{\beta\alpha}$ with $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = -\delta^\alpha_\gamma$.

The $OSp(2, 2|4)$ Poincaré supercharges are

$$Q^+_{a, \alpha} = q_{a, \alpha} + \epsilon_{\alpha\beta}\omega_{ab}\bar{q}^{b, \beta} = q_{a, \alpha} + \bar{q}_{a, \alpha}, \quad (2.16)$$

where $q_{a, \alpha}$ and $\bar{q}^{a, \dot{\alpha}}$ are the usual $\mathcal{N} = 4$ SYM Poincaré supercharges. Given a superconformal primary operator, we find their superconformal descendants by acting with Q^+ .

We now classify the representations of the operators appearing in our expansion and their multiplet structure.

2.3.1 Operators of classical dimension one

The only operator of classical dimension one in (2.11) is Φ^1 . This is an unprotected operator, so we will find its anomalous dimension below.

The other operators in the theory of classical dimension one are Φ^A , in the $(\mathbf{1}, \mathbf{5})$ of $SO(1, 2) \times SO(5)$. They are protected [16, 17] and do not mix with Φ^1 . They do not arise in our expansion, but as we show below, some of their descendents do.

⁸And $\bar{\sigma}^\mu = (1, -\tau^1, -\tau^3, -\tau^2)$.

2.3.2 Operators of classical dimension two

In (2.11) the operators

$$\{i\mathbb{F}_{i3}, \quad D_i\Phi^1, \quad iF_{ij}\}, \quad (2.17)$$

are of classical scaling dimension two.

The first two operators in (2.17) are not orthogonal,⁹ since $i\mathbb{F}_{i3} = iF_{i3} - D_i\Phi^1$. We therefore rewrite $D_i\Phi^1$ as a linear combination of $i\mathbb{F}_{i3}$ and its orthogonal operator $i\tilde{\mathbb{F}}_{i3} = iF_{i3} + 2D_i\Phi^1$

$$D_i\Phi^1 = \frac{1}{3} \left(i\tilde{\mathbb{F}}_{i3} - i\mathbb{F}_{i3} \right). \quad (2.18)$$

$i\mathbb{F}_{i3}$ is in the protected Φ^A supermultiplet, which is seen as follows. Acting on the primary Φ^A with the supercharges (2.16) we find

$$Q_{a,\alpha}^+ \Phi^A = i\bar{\rho}_{ab}^A \lambda_{\alpha}^{-,b}, \quad (2.19)$$

where $\lambda_{\alpha}^{\pm,a} = \lambda_{\alpha}^a \pm \bar{\lambda}_{\alpha}^a$ are the linear combinations of SYM fermions λ_{α}^a and $\bar{\lambda}_{\alpha}^{\dot{a}}$. Acting again with Q^+ gives

$$Q_{a,\alpha}^+ \lambda_{\beta}^{-,b} = 2i\delta^b{}_a \bar{\sigma}_{\alpha\beta}^i \mathbb{F}_{i3} + 2i\epsilon_{\alpha\beta} \omega^{bc} \bar{\rho}_{ca}^A \mathbb{D}_3 \Phi^A, \quad (2.20)$$

where the covariant derivative \mathbb{D}_3 is defined with respect to this modified connection \mathbb{A} . Since the two operators in (2.20) have different quantum numbers, we see that $i\mathbb{F}_{i3}$ is a superconformal descendant of Φ^A (and a conformal primary). The conformal descendant of Φ^A of scaling dimension two is the second operator in (2.20), $\mathbb{D}_3\Phi^A$, which does not appear in (2.17).

iF_{ij} is also in the **3** of $SO(1,2)$ which is in the Φ^1 supermultiplet. Acting with Q^+ on Φ^1 gives

$$Q_{a,\alpha}^+ \Phi^1 = i\lambda_{a,\alpha}^+, \quad (2.21)$$

and acting again with Q^+ gives

$$Q_{a,\alpha}^+ \lambda_{\beta}^{+,b} = 2i\epsilon_{\alpha\beta} \delta^b{}_a \mathbb{D}_3 \Phi^1 - 2\omega^{bc} \bar{\rho}_{ca}^A \bar{\sigma}_{\alpha\beta}^i D_i \Phi^A + (\sigma^{ij})_{\beta}{}^{\gamma} F_{ij} \epsilon_{\alpha\gamma} \delta^b{}_a + i\epsilon_{\alpha\beta} \rho^{A,bc} \bar{\rho}_{ca}^B [\Phi^A, \Phi^B]. \quad (2.22)$$

Projecting onto the $(\mathbf{3}, \mathbf{1})$ of $SO(1,2) \times SO(5)$ gives F_{ij} .

Finally $i\tilde{\mathbb{F}}_{i3}$ is in a third supermultiplet, to which it is the superconformal primary.

The other possible insertions of classical dimension two are

$$(\Phi^1)^2, \quad \Phi^1\Phi^A, \quad \Phi^A\Phi^1, \quad \Phi^A\Phi^B, \quad D_i\Phi^A, \quad \mathbb{D}_3\Phi^A, \quad \mathbb{D}_3\Phi^1, \quad (2.23)$$

whose quantum numbers are easy to read ($\Phi^A\Phi^B$ is comprised of the singlet, **10** and **14** of $SO(5)$). The three operators of dimension two appearing in the operator expansion (2.17) are in the $(\mathbf{3}, \mathbf{1})$ representation of $SO(1,2) \times SO(5)$. Clearly none of the operators above are in this representation. Thus, these operators may not mix with those appearing in the operator expansion.

⁹Here orthogonality refers to the two-point functions.

2.3.3 Operators of classical dimension three and four

The operators of classical scaling dimension three in (2.11), which arise in the expansion are

$$\{iD_i\mathbb{F}_{j3}, \quad iD_iF_{jk}\}, \quad (2.24)$$

Both of the operators above are in reducible representations of $SO(1,2) \times SO(5)$. The former reduces to the trace $(\mathbf{1}, \mathbf{1})$ and the traceless symmetric $(\mathbf{5}, \mathbf{1})$. The latter reduces to the trace $(\mathbf{3}, \mathbf{1})$ and the traceless symmetric $(\mathbf{5}, \mathbf{1})$.

$iD_{\{i}\mathbb{F}_{j\}3}$ is in the Φ^1 supermultiplet. Projecting onto the $(\mathbf{5}, \mathbf{1})$ of $SO(1,2) \times SO(5)$, we have

$$\epsilon^{abcd}\bar{\sigma}_{\{i}^{\alpha\beta}\bar{\sigma}_{j\}^{\gamma\delta}Q_{a,\alpha}^+Q_{b,\beta}^+Q_{c,\gamma}^+Q_{d,\delta}^+\Phi^1 = -1024D_{\{i}\mathbb{F}_{j\}3}. \quad (2.25)$$

The three other operators are neither primaries nor descendants. The new ingredient that arises for operators of dimension three is mixing with fermion bilinears and indeed the two traces of the operators in (2.24) mix with fermions to form (super-)descendants of Φ^1 and Φ^A .

Consider the equations of motion

$$\begin{aligned} iD^i\mathbb{F}_{i3} &= D^3D_3\Phi^1 + [D_3\Phi^1, \Phi^1] + [D_3\Phi^A, \Phi^A] + [[\Phi^1, \Phi^A], \Phi^A] + \frac{1}{2}[\lambda_{a,\alpha}^+, \lambda^{-a,\alpha}] \\ &= -\mathbb{D}_3\mathbb{D}_3\Phi^1 + [\mathbb{D}_3\Phi^A, \Phi^A] + \frac{1}{2}[\lambda_{a,\alpha}^+, \lambda^{-a,\alpha}], \end{aligned} \quad (2.26)$$

which shows that $iD^i\mathbb{F}_{i3}$ mixes with fermion bilinears (and the other commutator) to form a descendant of Φ^1 . We have not analyzed the full set of operators of this dimension to find the other combinations orthogonal to $\mathbb{D}_3\mathbb{D}_3\Phi^1$. In any case, for our two-loop analysis below, the mixing with the bilinears is suppressed, so to this order it is effectively a descendant.

For iD^iF_{ij} , we see from the equations of motion that

$$iD^iF_{ij} = -\frac{1}{3}\mathbb{D}_3\left(2i\mathbb{F}_{j3} + i\tilde{\mathbb{F}}_{j3}\right) + [\Phi^1, i\mathbb{F}_{j3}] + [D_j\Phi^A, \Phi^A] + \frac{1}{2}\sigma_j^{\alpha\beta}[\lambda_{a,\alpha}^+, \lambda_{\beta}^{a,-}], \quad (2.27)$$

so it is an admixture of descendants in two different supermultiplets. The final operator $iD_{\{i}F_{j\}k}$ is in neither the Φ^A nor the Φ^1 supermultiplet and does not appear in the operator expansion.

The only operators of dimension four which contribute to the expectation value of the deformed line at $\mathcal{O}(\epsilon^4)$ (see (3.38) below) are

$$\{iD^nD_{(n}\mathbb{F}_{m)3}\eta_{jk} + iD^nD_{(n}\mathbb{F}_{k)3}\eta_{jm} - \frac{2}{3}iD^nD_{(n}\mathbb{F}_{j)3}\eta_{km}, \quad iD_jD^n\mathbb{F}_{n3}\eta_{km}\}. \quad (2.28)$$

Up to mixing with bilinears, these are descendants of $i\mathbb{F}_{i3}$ *c.f.* (2.26).

3 Extracting the CFT data

We now use the equivalence between the operator insertions and small deformations of the 1/2 BPS Wilson line to extract the anomalous dimensions and two and three-point function

| Rep | Operator | Super-multiplet |
|-----------------------|---|----------------------|
| (1, 1, 1) | Φ^1 | Φ^1 |
| (1, 1, 5) | Φ^A | Φ^A |
| $(\frac{3}{2}, 2, 4)$ | $\lambda_{a\alpha}^-$ | Φ^A |
| | $\lambda_{a\alpha}^+$ | Φ^1 |
| (2, 1, 1) | $\mathbb{D}_3\Phi^1$ | Φ^1 |
| (2, 1, 5) | $\mathbb{D}_3\Phi^A$ | Φ^A |
| (2, 1, 10) | $[\Phi^A, \Phi^B]$ | Φ^1 |
| (2, 3, 1) | \mathbb{F}_{i3} | Φ^A |
| | F_{ij} | Φ^1 |
| | $\tilde{\mathbb{F}}_{i3}$ | $\tilde{\mathbb{F}}$ |
| (2, 3, 5) | $D_i\Phi^A$ | Φ^1 |
| $(\frac{5}{2}, 2, 4)$ | $\mathbb{D}_3\lambda_{a\alpha}^-$ | Φ^A |
| | $\mathbb{D}_3\lambda_{a\alpha}^+$ | Φ^1 |
| (3, 1, 1) | $D^i\mathbb{F}_{i3} + \dots$ | Φ^1 |
| (3, 1, 5) | $\mathbb{D}_3\mathbb{D}_3\Phi^A$ | Φ^A |
| (3, 3, 1) | $\mathbb{D}_3\mathbb{F}_{i3}$ | Φ^A |
| | $D^i F_{ij} + \frac{2}{3}\mathbb{D}_3\mathbb{F}_{j3} + \dots$ | $\tilde{\mathbb{F}}$ |
| | $D_{\{i}\mathbb{F}_{j\}3}$ | Φ^1 |

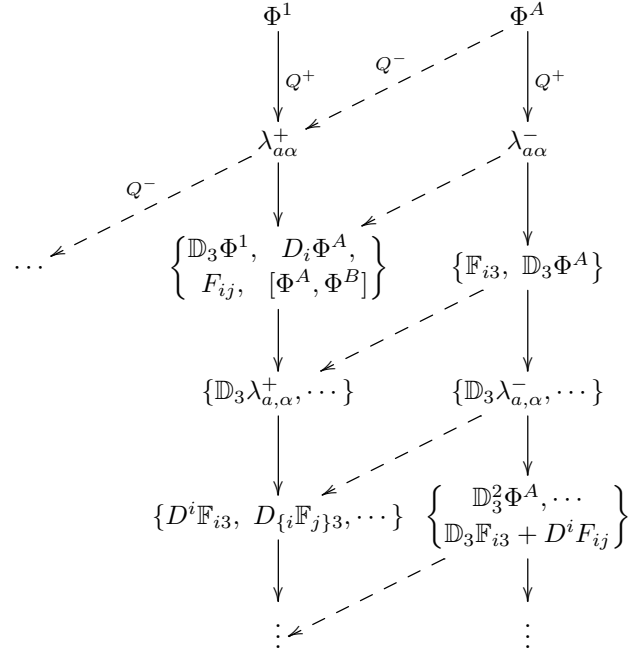


Figure 2: The insertions of lowest dimensions with their quantum numbers under $SO(1,2)^2 \times Sp(4)$ (the first is the dimension). The diagram shows the supermultiplets starting with the primary fields Φ^1 and Φ^A and acting with the unbroken supercharges $Q_{\alpha,a}^+$ to generate the super-descendants. The dashed arrows show the action of the broken supercharges $Q_{\alpha,a}^-$, which take us from one supermultiplet to another.

coefficients at weak coupling. The idea is to use the known perturbative expressions for a general Wilson loop at one and two-loops, expand these expressions in the deformation parameter ϵ and rewrite them in the form of integrated correlation functions of insertions.

At one-loop the Wilson loop is sensitive to two-point functions only. At two-loops things get more interesting, the Wilson loop is now sensitive to two, three and four-point functions. Beyond this simple statement, which is evident from the number of integrals in the perturbative expressions below (3.1), (3.2), there is a relation between the loop order in the Wilson loop evaluation and the CFT data. The one-loop expansion of the Wilson loop is only sensitive to the *classical* dimension of the operators and their *classical* two-point function coefficients. The two-loop expansion of the Wilson loop supplies the *one-loop* anomalous dimension and

coefficients and the *classical* structure constant. The four-point functions at this loop order is factorized and supplies no new information, just the classical dimensions already found before.

A higher loop analysis using this method would require expressions for generic Wilson loops, which are not available at present. It should be possible using our present technology to go to higher orders in ϵ , though issues of mixing of operators could arise.

We comment on the feasibility of extracting information from the holographic dual at strong coupling in Section 4.

3.1 The one and two-loop expectation values for general contours

At one-loop, the expectation value of a Wilson loop along a general smooth contour is given by the well known expression which with the mostly negative signature is

$$\langle W[\mathcal{C}] \rangle_{1\text{-loop}} = -\frac{\lambda}{16\pi^2} \oint ds_1 ds_2 I(s_1, s_2), \quad I(s_1, s_2) = \frac{\dot{x}_1 \cdot \dot{x}_2 + |\dot{x}_1| |\dot{x}_2|}{x_{12}^2}. \quad (3.1)$$

At two-loops there is an analogous expression for a general smooth contour with a constant scalar coupling. The sum of all two-loop Feynman diagrams in the planar approximation can be combined to the elegant expression [18]

$$\begin{aligned} \langle W[\mathcal{C}] \rangle_{2\text{-loop}} = & -\frac{\lambda^2}{128\pi^4} \oint ds_1 ds_2 ds_3 \epsilon(s_1, s_2, s_3) I(s_1, s_3) \frac{x_{32} \cdot \dot{x}_2}{x_{32}^2} \log \frac{x_{21}^2}{x_{31}^2} \\ & + \frac{\lambda^2}{2} \left(\frac{1}{16\pi^2} \oint ds_1 ds_2 I(s_1, s_2) \right)^2 - \frac{\lambda^2}{64\pi^4} \int_{s_1 > s_2 > s_3 > s_4} ds_1 ds_2 ds_3 ds_4 I(s_1, s_3) I(s_2, s_4). \end{aligned} \quad (3.2)$$

where $\epsilon(s_1, s_2, s_3)$ is completely antisymmetric and takes the value 1 for $s_1 > s_2 > s_3$.

3.1.1 Divergences and regularization

The one and two-loop integrals above are finite. However if we split the one-loop integral as

$$-\frac{\lambda}{8\pi^2} \int_{s_1 > s_2} ds_1 ds_2 \frac{\dot{x}_1 \cdot \dot{x}_2 + |\dot{x}_1| |\dot{x}_2|}{x_{12}^2} = -\frac{\lambda}{8\pi^2} \left(\int_{s_1 > s_2} ds_1 ds_2 \frac{\dot{x}_1 \cdot \dot{x}_2}{x_{12}^2} + \int_{s_1 > s_2} ds_1 ds_2 \frac{|\dot{x}_1| |\dot{x}_2|}{x_{12}^2} \right), \quad (3.3)$$

then each integral is separately divergent. The same is true for the two-loop integrals.

This is indeed the procedure we follow, regularizing each of the integrals independently and rewriting it in the form of integrated n -point functions. Within these final expressions we identify the regularization independent quantities, like the anomalous dimensions. For example, the last term in (3.3) can be integrated twice by parts to give a denominator of x_{12}^4 , which corresponds to the two-point function of an operator of dimension two.

We use two different regularization prescriptions. The first, point-splitting, puts a hard cutoff on the integrals above, such that the range of integration is $s_1 > s_2 + \mu$, and likewise cutoffs

at infinity. At two-loops the triple and quadruple integrals involve more cutoffs. Boundary terms expanded around the cutoff lead to many regularization dependant terms and overall rather messy expressions.

The other regularization we employ adds “mass terms”, or $i\varepsilon$ terms to the denominators, so $x_{12} \rightarrow x_{12} + i\zeta$. The resulting expressions are a bit cleaner and involve fewer regularization dependant terms.

3.2 Order ϵ^2

At order $\mathcal{O}(\epsilon)$, equation (2.5) includes only the one-point function of $i\mathbb{F}_{i3}$. This vanishes as discussed in the introduction, and any divergences that may appear are safely removed. We therefore start our discussion at order ϵ^2 , also known as the wavy line approximation, which is well studied and somewhat special [16,17]. We now rederive the known results to illustrate our approach.

Denoting by $\langle W[\mathcal{C}] \rangle|_{\epsilon^2}$ the contributions of order $\mathcal{O}(\epsilon^2)$ to the expectation value $\langle W[\mathcal{C}] \rangle$, the relevant terms from the expansion (2.5), (2.6) with the explicit operators (2.11) are

$$\begin{aligned} \langle W[\mathcal{C}] \rangle|_{\epsilon^2} = & \frac{1}{2!} \oint ds_1 \left(\epsilon_1^i \epsilon_1^j \langle\langle iD_{\{i\mathbb{F}_j\}3}(s_1)\rangle\rangle - \frac{1}{3} \delta_{ij} \epsilon_1^i \epsilon_1^j \langle\langle iD^i\mathbb{F}_{i3}(s_1)\rangle\rangle \right) \\ & + \frac{1}{2!} \oint ds_1 ds_2 \epsilon_1^i \epsilon_2^j \langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) \rangle\rangle, \end{aligned} \quad (3.4)$$

where $\epsilon_1^i = \epsilon^i(s_1)$ and similarly for s_2 . The two-point functions are constrained (since $i\mathbb{F}_{i3}$ is a conformal primary) to take the form

$$\langle W[\mathcal{C}] \rangle|_{\epsilon^2} = \int_{s_1 > s_2} ds_1 ds_2 \frac{a_F(\lambda)}{s_{12}^{2\Delta_F(\lambda)}} \eta_{ij} \epsilon_1^i \epsilon_2^j, \quad (3.5)$$

where $a_F(\lambda)$ is the two-point coefficient of $i\mathbb{F}_{i3}$ and $\Delta_F(\lambda)$ is its scaling dimension. (3.5) is true for all values of the coupling.

3.2.1 One-loop

To match with the perturbative expansion of the Wilson loop, we expand the all-coupling expression (3.5) at weak coupling. The $\mathcal{O}(\lambda)$ term is

$$\langle W[\mathcal{C}] \rangle_{1\text{-loop}}|_{\epsilon^2} = \lambda \int_{s_1 > s_2} ds_1 ds_2 \frac{a_F^0}{s_{12}^{2\Delta_F^0}} \eta_{ij} \epsilon_1^i \epsilon_2^j, \quad (3.6)$$

where a_F^0 is the tree-level two-point function coefficient and Δ_F^0 is the classical scaling dimension. In this case clearly $\Delta_F^0 = 2$, and it thus remains to determine a_F^0 .

To do this, we compare the above expression with (3.1) for the contour \mathcal{C} defined by $x^\mu = (\epsilon^i(s), s)$ and expand to order $\mathcal{O}(\epsilon^2)$

$$\langle W[\mathcal{C}] \rangle_{1\text{-loop}}|_{\epsilon^2} = -\frac{\lambda}{16\pi^2} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1} ds_2 \frac{(\dot{\epsilon}_1 - \dot{\epsilon}_2)^2}{s_{12}^2}. \quad (3.7)$$

We now split this integral into a sum of three integrals, each of which is divergent, so we need to introduce a regularization scheme. With a cutoff parameter μ we have

$$\begin{aligned} \langle W[\mathcal{C}] \rangle_{1\text{-loop}}|_{\epsilon^2} = & -\frac{\lambda}{16\pi^2} \left(\int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu} ds_2 \frac{\dot{\epsilon}_1^2}{s_{12}^2} - 2 \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu} ds_2 \frac{\dot{\epsilon}_1 \cdot \dot{\epsilon}_2}{s_{12}^2} \right. \\ & \left. + \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu} ds_2 \frac{\dot{\epsilon}_2^2}{s_{12}^2} \right), \end{aligned} \quad (3.8)$$

The first and last terms are symmetric with respect to $s_1 \leftrightarrow s_2$. We may perform the integration in s_2 and s_1 for these, respectively, without any knowledge of $\epsilon^i(s)$. We also integrate the second term by parts to find

$$\begin{aligned} \langle W[\mathcal{C}] \rangle_{1\text{-loop}}|_{\epsilon^2} = & -\frac{3\lambda}{4\pi^2} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu} ds_2 \frac{\epsilon_1 \cdot \epsilon_2}{s_{12}^4} - \frac{\lambda}{8\pi^2} \int_{-\infty}^{\infty} ds_1 \frac{\dot{\epsilon}_1^2}{\mu} \\ & + \frac{\lambda}{8\pi^2} \int_{-\infty}^{\infty} ds_1 \frac{\dot{\epsilon}(s_1) \cdot \epsilon(s_1 - \mu)}{\mu^2} + \frac{\lambda}{4\pi^2} \int_{-\infty}^{\infty} ds_1 \frac{\epsilon(s_1 + \mu) \cdot \epsilon(s_1)}{\mu^3}. \end{aligned} \quad (3.9)$$

The terms with single integrals are all one-point functions that can be discarded and we are left with

$$\langle W[\mathcal{C}] \rangle_{1\text{-loop}}|_{\epsilon^2} = -\frac{3\lambda}{4\pi^2} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu} ds_2 \frac{\epsilon_1 \cdot \epsilon_2}{s_{12}^4}. \quad (3.10)$$

We get the same result by employing mass regularization replacing s_{12} in (3.7) with $s_{12} + i\zeta_{12}$ giving

$$\begin{aligned} \langle W[\mathcal{C}] \rangle_{1\text{-loop}}|_{\epsilon^2} = & -\frac{\lambda}{16\pi^2} \left(\int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1} ds_2 \frac{\dot{\epsilon}_1^2}{(s_{12} + i\zeta_{12})^2} - 2 \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1} ds_2 \frac{\dot{\epsilon}_1 \cdot \dot{\epsilon}_2}{(s_{12} + i\zeta_{12})^2} \right. \\ & \left. + \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1} ds_2 \frac{\dot{\epsilon}_2^2}{(s_{12} + i\zeta_{12})^2} \right). \end{aligned} \quad (3.11)$$

Again we can integrate the first and last terms with respect to s_2 and s_3 respectively and integrate the second one twice by parts to find

$$\langle W[\mathcal{C}] \rangle_{1\text{-loop}}|_{\epsilon^2} = \frac{i\lambda}{16\pi^2} \oint ds_1 \left(\frac{\dot{\epsilon}_1^2}{\zeta_{12}} + \frac{4\epsilon_1^2}{\zeta_{12}^3} \right) - \frac{3\lambda}{4\pi^2} \int_{s_1 > s_2} ds_1 ds_2 \frac{\epsilon_1 \cdot \epsilon_2}{(s_{12} + i\zeta_{12})^4}, \quad (3.12)$$

where we have thrown away a total derivative. Discarding the single integral terms, reproduces (3.10).

Comparing this with the corresponding terms in (3.4), this is the tree-level contribution of the two-point of $i\mathbb{F}_{i3}$, from which we may extract the tree-level two-point function coefficient

$$a_F^0 = -\frac{3}{4\pi^2}. \quad (3.13)$$

This agrees with the expression found in [17], up to an overall sign, which is due to our choice of signature. The terms which were discarded are regularization artifacts which contain no physical information.

3.2.2 Two-loop

At order $\mathcal{O}(\lambda^2)$, equation (3.5) is

$$\langle W[\mathcal{C}] \rangle_{2\text{-loop}}|_{\epsilon^2} = \lambda^2 \int_{s_1 > s_2} ds_1 ds_2 \frac{a_F^1 - 2a_F^0 \gamma_F \log \frac{s_{12}}{m}}{s_{12}^{2\Delta_F^0}} \eta_{ij} \epsilon_1^i \epsilon_2^j, \quad (3.14)$$

where a_F^1 is the one-loop two-point function coefficient, γ_F is the anomalous dimension and m is a scale parameter. Clearly if the anomalous dimension is nonzero, then a_F^1 can be modified by changing m , and therefore is scheme dependent. As shown below, this is not the case for this operator, but is in fact true for most of the other operators we encounter. Expanding (3.2) to second order in ϵ gives

$$\langle W[\mathcal{C}] \rangle_{2\text{-loop}}|_{\epsilon^2} = \frac{\lambda^2}{256\pi^4} \oint ds_1 ds_2 ds_3 \epsilon(s_1, s_2, s_3) \frac{(\dot{\epsilon}_1 - \dot{\epsilon}_3)^2 \log \frac{s_{12}^2}{s_{13}^2}}{s_{13}^2 s_{23}}. \quad (3.15)$$

We restrict the domain of integration to $s_1 > s_2 > s_3$ and symmetrize the integrand accordingly. We also introduce two independent cutoffs μ_1 and μ_2

$$\begin{aligned} \langle W[\mathcal{C}] \rangle_{2\text{-loop}}|_{\epsilon^2} = \frac{\lambda^2}{256\pi^4} & \left(\int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu_1} ds_2 \int_{-\infty}^{s_2-\mu_2} ds_3 \frac{\dot{\epsilon}_1^2 \log \frac{s_{12}^2}{s_{13}^2}}{s_{13}^2 s_{23}} \right. \\ & + \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu_1} ds_2 \int_{-\infty}^{s_2-\mu_2} ds_3 \frac{\dot{\epsilon}_3^2 \log \frac{s_{12}^2}{s_{13}^2}}{s_{13}^2 s_{23}} \\ & \left. - 2 \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu_1} ds_2 \int_{-\infty}^{s_2-\mu_2} ds_3 \frac{\dot{\epsilon}_1 \cdot \dot{\epsilon}_3 \log \frac{s_{12}^2}{s_{13}^2}}{s_{13}^2 s_{23}} \right) \\ & + (\text{symmetrization}). \end{aligned} \quad (3.16)$$

Similarly to the one-loop case, we integrate the first term with respect to s_2 and s_3 and the second term with respect to s_1 and s_2 , leading to divergent one-point function expressions which can be omitted. For the third term we may perform the s_2 integral to get

$$- \frac{\lambda^2}{64\pi^4} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu_1-\mu_2} ds_3 \frac{\dot{\epsilon}_1 \cdot \dot{\epsilon}_3}{s_{13}^2} \left(\text{Li}_2 \left(\frac{\mu_2}{s_{13}} \right) - \text{Li}_2 \left(1 - \frac{\mu_1}{s_{13}} \right) \right) \quad (3.17)$$

We find here two terms, one dependent on μ_2 and the other on μ_1 . Focusing on the first, it is possible to shift s_1 by μ_2 such that μ_2 appears only in the integrand and μ_1 only in the range of s_3 . This clear separation of the two cutoffs simplifies the calculation of the divergent terms and is the procedure we employ for subsequent multi-integrals in the following.

For the case at hand, these procedures are not strictly necessary, as the integrand has a simple limit for $\mu_1, \mu_2 \rightarrow 0$, namely $\text{Li}_2(0) = 0$ and $\text{Li}_2(1) = \frac{\pi^2}{6}$. One then integrates by parts with respect to both s_1 and s_3 , and the boundary terms can be safely ignored.¹⁰ We find

$$-\frac{\lambda^2}{64\pi^4} \int_{s_1 < s_3} ds_1 ds_3 \epsilon_1 \cdot \epsilon_3 \frac{d^2}{ds_1 ds_3} \frac{-\pi^2}{6s_{13}^2} = -\frac{\lambda^2}{64\pi^2} \int_{s_1 < s_3} ds_1 ds_3 \frac{\epsilon_1 \cdot \epsilon_3}{s_{13}^4}. \quad (3.18)$$

Under the symmetrization there is another term with $\dot{\epsilon}_1 \cdot \dot{\epsilon}_3$, so again we can do the s_2 integral, which ends up giving an identical contribution. Then there are four other terms with s_2 exchanged for s_1 or s_3 , whose contribution is minus the above. All together we find

$$\langle W[\mathcal{C}] \rangle_{2\text{-loop}}|_{\epsilon^2} = \frac{\lambda^2}{32\pi^2} \int_{s_1 > s_2} ds_1 ds_2 \frac{\epsilon_1 \cdot \epsilon_2}{s_{12}^4}. \quad (3.19)$$

We can also perform the integrals in (3.15) in the other regularization scheme, where it now takes the form

$$\langle W[\mathcal{C}] \rangle_{2\text{-loop}}|_{\epsilon^2} = \frac{\lambda^2}{256\pi^4} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1} ds_2 \int_{-\infty}^{s_2} ds_3 \frac{(\dot{\epsilon}_1 - \dot{\epsilon}_3)^2 \log \frac{(s_{12} + i\zeta_{12})^2}{(s_{13} + i\zeta_{13})^2}}{(s_{13} + i\zeta_{13})^2 (s_{23} + i\zeta_{23})} + (\text{sym}). \quad (3.20)$$

Splitting as above, the integrals of the terms analogous to the first two lines of (3.16) give poles in the ζ_{ij} parameters and the third line gives again an expression which reduces to (3.17) in the $\zeta_{13} \rightarrow 0$ limit (and one may also take this limit after the integration by parts), so leading to the same result.

Comparing with (3.14) gives

$$a_F^1 = \frac{1}{32\pi^2}, \quad \gamma_F = 0, \quad (3.21)$$

so

$$a_F = -\frac{3\lambda}{4\pi^2} + \frac{\lambda^2}{32\pi^2} + O(\lambda^3). \quad (3.22)$$

This matches the results of [17] up to the difference in sign mentioned above and is consistent with the fact that $i\mathbb{F}_{i3}$ is a protected operator with vanishing anomalous dimension. This is also the origin of the “universality” property, where at all values of the coupling

$$\langle W[\mathcal{C}] \rangle|_{\epsilon^2} = \oint ds_1 ds_2 \frac{a_F(\lambda) \eta_{ij} \epsilon^i \epsilon^j}{s_{12}^4}, \quad (3.23)$$

¹⁰As presented they are all divergent, but if we keep subleading terms in the expansions of Li_2 , there may also be finite terms. Still those are one-point functions which should vanish, and indeed we find that they are all total derivatives which can be integrated away.

and the only dependence on the coupling is in $a_F(\lambda)$. And indeed this also can be derived at strong coupling from the AdS dual, where $a_F = -3\sqrt{\lambda}/\pi^2$ [17]. The exact result is given by $a_F = -12B(\lambda)$, where $B(\lambda) = \frac{\sqrt{\lambda}I_2(\sqrt{\lambda})}{4\pi^2 I_1(\sqrt{\lambda})}$ is the Bremsstrahlung function [16] ($I_n(x)$ are modified Bessel functions).

3.3 Order ϵ^3

At $\mathcal{O}(\epsilon^3)$ we have the expression in (2.7) with the explicit operators (2.11) which depends on the following two and three-point functions

$$\begin{aligned} &\langle\langle i\mathbb{F}_{i3}(s_1)iD^j\mathbb{F}_{j3}(s_2)\rangle\rangle, \quad \langle\langle i\mathbb{F}_{i3}(s_1)iD_{\{j\mathbb{F}_k\}3}(s_2)\rangle\rangle, \quad \langle\langle i\mathbb{F}_{i3}(s_1)iF_{jk}(s_2)\rangle\rangle, \\ &\langle\langle i\mathbb{F}_{i3}(s_1)\Phi^1(s_2)\rangle\rangle, \quad \langle\langle i\mathbb{F}_{i3}(s_1)i\mathbb{F}_{j3}(s_2)i\mathbb{F}_{k3}(s_3)\rangle\rangle. \end{aligned} \quad (3.24)$$

Comparing with the table of operators in Figure 2, we see that all two-point functions involve pairs of operators from different supermultiplets, and we would therefore expect them to cancel. The last expression, which is the three-point function of \mathbb{F}_{i3} also vanishes, because the only invariant 3-tensor is antisymmetric.

Indeed the perturbative expressions at one-loop (3.1) and two-loop (3.2) do not contain any terms at $\mathcal{O}(\epsilon^3)$.

3.4 Order ϵ^4

At $\mathcal{O}(\epsilon^4)$ there are contributions from two, three and four-point functions. The operator expansion at this order is quite long and messy. As such we deal with the four, three and two-point functions separately. We further analyze the two-point functions of the different multiplets separately.

3.4.1 The four-point functions

The sole four-point function appearing in the operator expansion (2.8) at $\mathcal{O}(\epsilon^4)$ is

$$\int_{s_1 > s_2 > s_3 > s_4} ds_1 ds_2 ds_3 ds_4 \langle\langle i\mathbb{F}_{i3}(s_1)i\mathbb{F}_{j3}(s_2)i\mathbb{F}_{k3}(s_3)i\mathbb{F}_{m3}(s_4)\rangle\rangle \epsilon_1^i \epsilon_2^j \epsilon_3^k \epsilon_4^m. \quad (3.25)$$

At two-loops we have the quadruple integrals on the second line of (3.2), which involve two of the one-loop generalized propagator (3.1) and the dependence on the deformation is $\epsilon_1^i \epsilon_2^j \epsilon_3^k \epsilon_4^m$. As in the one-loop loop calculation in Section 3.2.1, we can perform multiple integration by parts, to reproduce the deformation dependence as in (3.25), with the two-point functions as calculated in Section 3.2.1.

$$-\lambda^2 \int_{s_1 > s_2 > s_3 > s_4} ds_1 ds_2 ds_3 ds_4 \frac{a_F^0}{s_{13}^2 \Delta_F^0} \frac{a_F^0}{s_{24}^2 \Delta_F^0} (\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4) + \frac{\lambda^2}{2} \left(\oint ds_1 ds_2 \frac{a_F^0}{s_{13}^2 \Delta_F^0} \epsilon_1 \cdot \epsilon_2 \right)^2. \quad (3.26)$$

This is exactly the factorized form of the four-point function in (3.25), as would be expected at two-loop order.

Thus this calculation contains no new information, but rather serves as a consistency check of our approach.

3.4.2 The three-point functions

Using the expressions from Section 2 it is clear that at order ϵ^4 , the three-point functions that appear involve two insertions of $i\mathbb{F}_{i3}$ and one operator from $\mathcal{S}^{(2)}$ in (2.11). the three-point function contribution to the operator expansion is

$$\begin{aligned} \frac{1}{4} \oint ds_1 ds_2 ds_3 & \left(\langle\langle i\mathbb{F}_{i3}(s_1)i\mathbb{F}_{j3}(s_2)iD_{\{k}\mathbb{F}_m\}3} \rangle\rangle \epsilon_1^i \epsilon_2^j \epsilon_3^k \epsilon_3^m + \frac{1}{3} \langle\langle i\mathbb{F}_{i3}(s_1)i\mathbb{F}_{j3}(s_2)iD^m\mathbb{F}_{m3} \rangle\rangle \epsilon_1^i \epsilon_2^j \epsilon_3^k \epsilon_{3k} \right. \\ & \left. + \langle\langle i\mathbb{F}_{i3}(s_1)i\mathbb{F}_{j3}(s_2)iF_{km} \rangle\rangle \epsilon_1^i \epsilon_2^j \epsilon_3^k \epsilon_3^m + \langle\langle i\mathbb{F}_{i3}(s_1)i\mathbb{F}_{j3}(s_2)\Phi^1(s_3) \rangle\rangle \epsilon_1^i \epsilon_2^j \epsilon_3^k \epsilon_{3k} \right). \end{aligned} \quad (3.27)$$

Both $i\mathbb{F}_{i3}$ and Φ^1 are conformal primaries and applying the constraints of the residual conformal symmetry, as described in Section 1 we expect

$$\langle\langle i\mathbb{F}_{i3}(s_1)i\mathbb{F}_{j3}(s_2)\Phi^1(s_3) \rangle\rangle = \frac{c_\Phi^0 \eta_{ij}}{|s_{12}|^3 |s_{13}| |s_{23}|} + \mathcal{O}(\lambda). \quad (3.28)$$

We emphasize again that the three-point functions which appears at two-loop order in the expectation value of the deformed Wilson loop are the tree-level coefficients. We have seen in Section 2 that up to mixing with operators that do not contribute at this order, $iD^i\mathbb{F}_{i3}$ is a second level descendant of Φ^1 . Thus we expect

$$\langle\langle i\mathbb{F}_{i3}(s_1)i\mathbb{F}_{j3}(s_2)iD^k\mathbb{F}_{k3}(s_3) \rangle\rangle = -\frac{d^2}{ds_3^2} \langle\langle i\mathbb{F}_{i3}(s_1)i\mathbb{F}_{j3}(s_2)\Phi^1(s_3) \rangle\rangle, \quad (3.29)$$

which serves as another consistency check.

The remaining two operators above, iF_{ij} and $iD_{\{i}\mathbb{F}_j\}3}$, are conformal primaries, one in the **3** and one in the **5** or $SO(1,2)$ and the three-point functions involving these primaries have a complicated tensor structure, but we label the overall prefactors by $c_{\mathbf{3}}^0$ and $c_{\mathbf{5}}^0$.

To find these coefficients, we study the triple integrals in the two-loop expression of the Wilson loop (3.2) and expand the curve to fourth order in ϵ . In addition to the original triple integral on the first line of (3.2), there are boundary terms from integrating the quadruple integrals by parts. As discussed above at order ϵ^2 , the resulting expressions are divergent and careful regularization is required. After following the procedure outlined there, we find that

that tensor structures of the complicated three-point functions are¹¹

$$\begin{aligned}\langle\langle i\mathbb{F}_{i3}(s_1)i\mathbb{F}_{j3}(s_2)iF_{km}(s_3)\rangle\rangle &= \frac{c_{\mathbf{3}}^0 (\eta_{ik}\eta_{jm} - \eta_{im}\eta_{jk})}{|s_{12}|^2|s_{13}|^2|s_{23}|^2} + \mathcal{O}(\lambda), \\ \langle\langle i\mathbb{F}_{i3}(s_1)i\mathbb{F}_{j3}(s_2)iD_{\{k\mathbb{F}_m\}3}(s_3)\rangle\rangle &= \frac{c_{\mathbf{5}}^0 (\eta_{ik}\eta_{jm} + \eta_{im}\eta_{jk} - \frac{2}{3}\eta_{ij}\eta_{km})}{|s_{12}|^2|s_{13}|^2|s_{23}|^2} + \mathcal{O}(\lambda),\end{aligned}\tag{3.30}$$

The calculation also provides the values of the tree-level structure constants as

$$c_{\Phi}^0 = -\frac{1}{32\pi^4}, \quad c_{\mathbf{3}}^0 = \frac{1}{16\pi^4}, \quad c_{\mathbf{5}}^0 = \frac{5}{16\pi^4}.\tag{3.31}$$

The calculation also confirms that the descendanty condition (3.29) is satisfied.

3.4.3 Two-point functions

We now proceed to discuss the two-point functions. Here there are many terms and we write only those who do not vanish—those between operators in the same multiplets.

The information for the Φ^A supermultiplet could already be gleaned from the $O(\epsilon^2)$ analysis in Section 3.2. The new information is about the two-point functions of operators in the Φ^1 supermultiplet.

The terms in the operator expansion involving Φ^1 and its conformal descendants in the operator expansion are

$$\begin{aligned}\oint ds_1 ds_2 \Big(\frac{1}{72} \langle\langle iD^k\mathbb{F}_{k3}(s_1)iD^m\mathbb{F}_{m3}(s_2)\rangle\rangle \epsilon_1^i \epsilon_{1i} \epsilon_2^j \epsilon_{2j} + \frac{1}{8} \langle\langle \Phi^1(s_1)\Phi^1(s_2)\rangle\rangle \dot{\epsilon}_1^i \dot{\epsilon}_{1i} \dot{\epsilon}_2^j \dot{\epsilon}_{2j} \\ + \frac{1}{12} \langle\langle \Phi^1(s_1)iD^k\mathbb{F}_{k3}(s_2)\rangle\rangle \dot{\epsilon}_1^i \dot{\epsilon}_{1i} \epsilon_2^j \epsilon_{2j} \Big).\end{aligned}\tag{3.32}$$

The information we expect to extract is a_{Φ}^0 and γ_{Φ} , the tree-level two-point function coefficient of Φ^1 and its anomalous dimension. Since the latter does not vanish, the value of a_{Φ}^1 is scheme dependant, and indeed we could easily reproduce different values by changing details of our regularization procedures.

The calculation of the three and four-point functions above, which required integrating (3.2) by parts left many double integrals. In the resulting expressions we identified the terms proportional to the combinations of ϵ_1 and ϵ_2 in (3.32), leading to the values

$$a_{\Phi}^0 = \frac{1}{8\pi^2}, \quad \gamma_{\Phi} = \frac{1}{4\pi^2}.\tag{3.33}$$

The anomalous dimension matches that found in [19], where it was calculated from Feynman diagrams.

¹¹We verified the tensor structure of the first one independently from conformal symmetry, but not of the second one.

In this supermultiplet we also have the iF_{ij} multiplet, which clearly should have the same anomalous dimension, but possibly a different two-point function coefficient

$$\frac{1}{4} \int_{s_1 > s_2} ds_1 ds_2 \langle\langle iF_{ij}(s_1) iF_{km}(s_2) \rangle\rangle \epsilon_1^i \epsilon_1^j \epsilon_2^k \epsilon_2^m. \quad (3.34)$$

Indeed we find

$$a_{\mathbf{3}}^0 = -\frac{1}{2\pi^2}, \quad \gamma_{\mathbf{3}} = \frac{1}{4\pi^2}. \quad (3.35)$$

Similarly, we have

$$\int_{s_1 > s_2} ds_1 ds_2 \frac{1}{4} \langle\langle iD_{\{i\mathbb{F}_j\}3}(s_1) iD_{\{k\mathbb{F}_m\}3}(s_2) \rangle\rangle \epsilon_1^i \epsilon_1^j \epsilon_2^k \epsilon_2^m, \quad (3.36)$$

and we find

$$a_{\mathbf{5}}^0 = \frac{5}{\pi^2}, \quad \gamma_{\mathbf{5}} = \frac{1}{4\pi^2}. \quad (3.37)$$

For completeness, we include the terms at this order from the Φ^4 multiplet. Those are

$$\begin{aligned} & \oint ds_1 ds_2 \left(\frac{2}{45} \langle\langle i\mathbb{F}_{i3}(s_1) iD^n D_{(n\mathbb{F}_m)3}(s_2) \rangle\rangle \eta_{jk} \epsilon_1^i \epsilon_2^j \epsilon_2^k \epsilon_2^m \right. \\ & + \frac{1}{18} \langle\langle i\mathbb{F}_{i3}(s_1) iD_j D^n \mathbb{F}_{n3}(s_2) \rangle\rangle \eta_{km} \epsilon_1^i \epsilon_2^j \epsilon_2^k \epsilon_2^m - \frac{1}{6} \langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) \rangle\rangle \epsilon_1^i \epsilon_2^j \epsilon_2^k \epsilon_2^m \\ & \left. + \frac{1}{9} \langle\langle i\mathbb{F}_{i3}(s_1) iD^n F_{nm}(s_2) \rangle\rangle \eta_{jk} \epsilon_1^i \epsilon_2^j \epsilon_2^k \epsilon_2^m \right). \end{aligned} \quad (3.38)$$

All of the physical information in the correlation functions above should be fixed by the descendency relation from the expressions for \mathbb{F}_{i3} (3.22) in Section 3.2. When trying to identify these expressions in the Wilson loop expansion, we find that the anomalous dimension indeed vanishes as required. However, we find extra terms that do not match with the second order a_F^1 calculated above. In addition, in cutoff regularization we find also the terms

$$-\frac{1}{16\pi^4}(\dot{\epsilon}^1 \cdot \dot{\epsilon}^1)(\dot{\epsilon}^1 \cdot \epsilon^2) + \frac{1}{16\pi^4}(\epsilon^1 \cdot \dot{\epsilon}^2)(\dot{\epsilon}^2 \cdot \dot{\epsilon}^2) + \frac{1}{64\pi^4}(\epsilon^1 \cdot \dot{\epsilon}^1)(\ddot{\epsilon}^1 \cdot \epsilon^2) - \frac{1}{64\pi^4}(\epsilon^1 \cdot \ddot{\epsilon}^2)(\epsilon^2 \cdot \dot{\epsilon}^2) \quad (3.39)$$

that do not appear in any of the expression above. The coefficients of these unwanted terms (as well as the expression for a_F^1) depend on details of the regularization procedure, the exact choice of cutoffs and mass terms. This indicates some subtle inconsistencies in our regularization schemes (or computational errors), as $a_F^{(1)}$ should be well defined. Since there were no such problems in the calculation at order ϵ^2 , we expect that the results in Section 3.2 including the value of $a_F^{(1)}$ derived there are correct (and indeed they agree with [17]). Likewise all the quantities stated above, the tree-level two-point function coefficient and anomalous dimensions are robust in our calculation, as are the structure constants, and we expect them to be correct and scheme independent.

4 Discussion

In this note we studied the properties of operator insertions into the 1/2 BPS Wilson loop of $\mathcal{N} = 4$ SYM theory. The expectation value of the Wilson loop with insertions has the structure of a (defect) CFT and therefore operators can be assigned a normalization, conformal dimensions and structure constants.

To do this, we first classified operator insertions according to the residual symmetry group of the space-like Wilson line, $OSp(2, 2|4)$. We then considered the expectation value at one and two-loop order of smooth curves which are small deformations of the straight line. Expanding those in the deformation parameter and manipulating the integrals we found expressions resembling two, three and four-point functions of operators of different dimensions. We could then match these expressions to particular operator insertions, via the correspondence between small deformations of Wilson loops and operator insertions.

Some of the information we found has been known before: The dimension of Φ^4 and its descendants, which are protected, its one and two-loop two-point function coefficient, and the one-loop anomalous dimension of Φ^1 . Beyond that we found several more two-point function coefficients and several structure constants.

It may be possible to use our approach to study operators of higher dimensions by expanding to higher order in the deformation parameter ϵ . To go to higher orders in perturbation theory would require general formulae for the three-loop expectation value of a Wilson loop, which is far beyond current technology. At that order further issues, like operator mixing which did not effect our calculation, will arise. An alternative approach to this problem would employ integrability, which should rather easily provide anomalous dimensions to higher orders (by analytic perturbative calculation or numerical evaluation at finite coupling). Further developments would be required to find structure constants.

In the case of insertion along a Wilson loop with an arbitrary smooth contour, the loop breaks conformal invariance completely and one cannot define anomalous dimensions anymore. Still, the local analysis is identical, and since it is known that loops are renormalized in a multiplicative fashion, with a factor for each insertion (and cusp) [2–5], those renormalization factors can be taken from our analysis of the straight line.

One may also want to study these quantities at strong coupling, via the AdS/CFT correspondence. An efficient algorithm exists to calculate the expectation value of Wilson loops which are deformations of the 1/2 BPS circle (or line) to high orders in ϵ [23]. This is based on a rather general approach to the problem of minimal surfaces in AdS_3 [25], so the Wilson loops have to be restricted to $\mathbb{R}^2 \subset \mathbb{R}^{1,3}$. This means that the deformations ϵ^i have to be in a single direction normal to the line, which may lead to some confusions among different $SO(1, 2)$ tensors.

At order ϵ^2 this would reproduce the known large coupling behavior of the two-point function coefficient of the first insertion $i\mathbb{F}_{23}$. At order ϵ^4 there would be again several two, three and

four-point functions which would contribute to the calculation. In contrast to the weak coupling calculation presented here, the structure of the four-point function is not restricted to factorize. Rather it can take the form of (1.6) with an arbitrary function $G_{\text{FFFF}}(u, v)$. While a finite number of deformations can determine the two and three-point functions, an infinite number of deformations would be required to fix this. Still, since ϵ can be taken to be general, it may be possible to solve for this function.

At higher order in ϵ one would encounter five, six and higher point functions and the problem can get more complicated. Of course, one may try to ignore these issues, as well as that of the four-point function above, as knowledge of the two and three-point functions determines a CFT. So one can disregard the four and higher point function contributions (or match them as in the last paragraph, or using the OPE. We leave these issues to the future.

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References

- [1] A. M. Polyakov, “Gauge fields as rings of glue,” *Nucl.Phys.* **B164** (1980) 171–188.
- [2] R. A. Brandt, F. Neri, and M. Sato, “Renormalization of loop functions for all loops,” *Phys.Rev.* **D24** (1981) 879.
- [3] N. S. Craigie and H. Dorn, “On the renormalization and short distance properties of hadronic operators in QCD,” *Nucl. Phys.* **B185** (1981) 204–220.
- [4] R. A. Brandt, A. Gocksch, M. Sato, and F. Neri, “Loop space,” *Phys.Rev.* **D26** (1982) 3611.
- [5] H. Dorn, “Renormalization of path ordered phase factors and related hadron operators in gauge field theories,” *Fortsch. Phys.* **34** (1986) 11–56.
- [6] A. M. Polyakov and V. S. Rychkov, “Loop dynamics and AdS/CFT correspondence,” *Nucl. Phys.* **B594** (2001) 272–286, [hep-th/0005173](#).
- [7] A. M. Polyakov and V. S. Rychkov, “Gauge field strings duality and the loop equation,” *Nucl. Phys.* **B581** (2000) 116–134, [hep-th/0002106](#).
- [8] N. Drukker and S. Kawamoto, “Small deformations of supersymmetric Wilson loops and open spin-chains,” *JHEP* **07** (2006) 024, [hep-th/0604124](#).
- [9] N. Drukker, “Integrable Wilson loops,” *JHEP* **10** (2013) 135, [arXiv:1203.1617](#).

- [10] D. Correa, J. Maldacena, and A. Sever, “The quark anti-quark potential and the cusp anomalous dimension from a TBA equation,” *JHEP* **08** (2012) 134, [arXiv:1203.1913](#).
- [11] N. Gromov and F. Levkovich-Maslyuk, “Quark–anti-quark potential in $\mathcal{N} = 4$ SYM,” *JHEP* **12** (2016) 122, [arXiv:1601.05679](#).
- [12] J. L. Cardy, “Conformal invariance and surface critical behavior,” *Nucl. Phys.* **B240** (1984) 514–532.
- [13] D. M. McAvity and H. Osborn, “Conformal field theories near a boundary in general dimensions,” *Nucl. Phys.* **B455** (1995) 522–576, [cond-mat/9505127](#).
- [14] A. Karch and L. Randall, “Open and closed string interpretation of SUSY CFT’s on branes with boundaries,” *JHEP* **06** (2001) 063, [hep-th/0105132](#).
- [15] O. DeWolfe, D. Z. Freedman, and H. Ooguri, “Holography and defect conformal field theories,” *Phys. Rev.* **D66** (2002) 025009, [hep-th/0111135](#).
- [16] D. Correa, J. Henn, J. Maldacena, and A. Sever, “An exact formula for the radiation of a moving quark in $\mathcal{N} = 4$ super Yang Mills,” *JHEP* **06** (2012) 048, [arXiv:1202.4455](#).
- [17] G. W. Semenoff and D. Young, “Wavy Wilson line and AdS/CFT ,” *Int. J. Mod. Phys.* **A20** (2005) 2833–2846, [hep-th/0405288](#).
- [18] A. Bassetto, L. Griguolo, F. Pucci, and D. Seminara, “Supersymmetric Wilson loops at two loops,” *JHEP* **06** (2008) 083, [arXiv:0804.3973](#).
- [19] L. F. Alday and J. Maldacena, “Comments on gluon scattering amplitudes via AdS/CFT ,” *JHEP* **11** (2007) 068, [arXiv:0710.1060](#).
- [20] A. V. Belitsky, S. E. Derkachov, G. P. Korchemsky, and A. N. Manashov, “Superconformal operators in $\mathcal{N} = 4$ superYang-Mills theory,” *Phys. Rev.* **D70** (2004) 045021, [hep-th/0311104](#).
- [21] J. Groeger, *Supersymmetric Wilson loops in $\mathcal{N} = 4$ super Yang-Mills theory*. PhD thesis, Humboldt U., Berlin, 2012.
<http://inspirehep.net/record/1217936/files/DA-Groeger.pdf>.
- [22] K. Wiegandt, *Superconformal quantum field theories in string - gauge theory dualities*. PhD thesis, Humboldt U., Berlin, 2012.
<http://edoc.hu-berlin.de/browsing/dissertationen/>.
- [23] A. Dekel, “Wilson loops and minimal surfaces beyond the wavy approximation,” *JHEP* **03** (2015) 085, [arXiv:1501.04202](#).
- [24] M. Bianchi, M. B. Green, and S. Kovacs, “Instanton corrections to circular Wilson loops in $\mathcal{N} = 4$ supersymmetric Yang-Mills,” *JHEP* **04** (2002) 040, [hep-th/0202003](#).
- [25] M. Kruczenski, “Wilson loops and minimal area surfaces in hyperbolic space,” *JHEP* **11** (2014) 065, [arXiv:1406.4945](#).